

Lack of thermalization in holographic superconductivity

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It is expected that, after a quench, an interacting quantum system eventually reaches thermal equilibrium. Here we show that this is not always the case. We investigate the dynamics of the order parameter of a strongly coupled superconductor after a quench by holographic techniques. The gravity dual that we employ is the AdS_5 Soliton background at zero temperature. Time evolution is first investigated by computing the quasi normal modes (QNM) associated to the superconducting order parameter. QNM's are purely real which suggests undamped time oscillations of the order parameter which prevents thermalization of the dual field theory. Results for the time evolution of the order parameter after an abrupt change of the chemical potential are consistent with this picture. By tuning the quench strength we identify a region of parameters for which the oscillating pattern of the order parameter becomes increasingly intricate. We also discuss the range of applicability of these results and the physical origin of the lack of thermalization.

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Advances in cold atom physics and numerical techniques have renewed the interest in the route and conditions for thermalization in closed, strongly interacting, quantum systems [1, 2] after a quench. Until recently theoretical progress had been relatively slow as traditional analytical and numerical approaches are not in general well suited to describe far from equilibrium dynamics. However the application of the holographic principle, also called the AdS/CFT correspondence [3], has opened [4–11] new research avenues to tackle this problem. This correspondence states that, under certain conditions, strongly coupled gauge conformal field theories (CFT) in d dimensions are dual to a classical gravity theory in Anti de Sitter (AdS) space in $d+1$ dimensions. The out of equilibrium dynamics of the field theory has an especially appealing interpretation in the gravity dual: it corresponds with the time evolution of a mass distribution in an asymptotic AdS background. Thermalization is thus related to the formation of a black hole in the gravity dual [7, 8]. In the context of holographic superconductivity [12] there are recent studies on the time evolution of the order parameter [6, 9, 13, 14] at finite temperature. In [14] it was found that, in the probe limit, QNM of the order parameter have finite real and imaginary parts which indicates that, for long times, the superconducting order parameter oscillates with an amplitude that decays exponentially as thermal equilibrium is approached. Moreover Ref.[14] also identified the Goldstone mode that signals the spontaneous breaking of the $U(1)$ symmetry at sufficiently low temperature. These results have been recently confirmed in [9] by the explicit calculation of the order parameter after a quench taking into

account backreaction effects. Similarly a previous study [6], also including backreaction, showed that, for sufficiently long times after a quench, deviations of the order parameter from the thermal equilibrium prediction are exponentially small. Moreover a quench in a Reissner-Nordström-AdS black hole can induce an instability in the metric that leads to holographic superconductivity [6].

Recent experimental [2] and theoretical [1, 15, 16] results in condensed matter points to a richer phenomenology. Integrability and localization [1, 2, 15] can prevent, or slow down [15], thermalization after a quench at zero temperature. In the context of weakly coupled superconductors [16] undamped oscillations of the order parameter after a quench have been observed provided that the final coupling is much larger than the initial one. For intermediate quenches the superconducting gap is oscillatory in time with an amplitude that decays towards equilibrium in a power-law fashion [16].

Motivated by these results we explore novel dynamical regions in the time evolution of holographic superconductors. Our main aim is to identify a range of parameters for which thermalization does not occur. For this purpose we employ a superconducting AdS Soliton background [17, 18] which has a well defined limit at zero temperature. This is a key departure from previous holographic studies [6, 9, 14] which were carried out at finite temperature, namely, in an AdS black hole background. At finite temperature thermalization always occurs after sufficiently long times. Therefore it is important to work in a background which has a well defined limit at zero temperature. In order to gain insight into the late

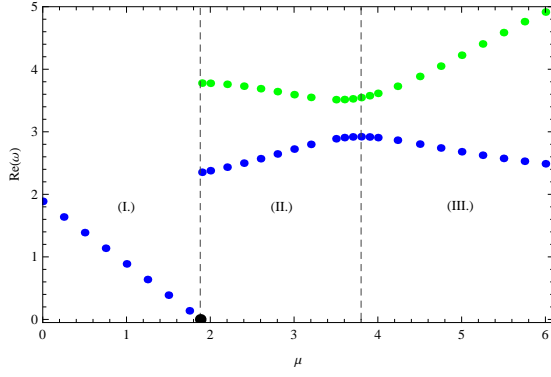


FIG. 1. (Color online) First (blue) and second (green) order QNM $\text{Re}[\omega]$ (9) as a function of μ . A vanishing QNM (black dot) signals the superconducting transition at $\mu = \mu_c \approx 1.88$ [17]. Within the numerical precision of the calculation the imaginary part of the QNM vanishes for any μ . This suggests that after a quench the dual field theory does not thermalize. The dynamics of the order parameter (see Fig. 3) becomes more intricate for $\mu \geq 3.8$ which corresponds to the minimum μ needed to excite the second order QNM.

time dynamics of the order parameter we study first the QNM [19] related to the order parameter [20]. We have found that in the AdS Soliton background the QNM's are always real which suggests that the order parameter oscillates without any damping. A detailed analysis of the time evolution after a quench confirms this prediction. Therefore no thermalization occurs, namely, after a quench the strongly coupled superconductor does not reach thermal equilibrium. However the oscillating pattern strongly depends on the quench details. In some cases we observe quasi chaotic oscillations.

The model.-

We study the time evolution of a holographic superconductor in an AdS_5 Soliton [17] gravity background. This background [18] is constructed from the usual AdS_5 geometry by a double Wick rotation followed by compactification of one of the spatial dimensions. In order to have a smooth geometry it is necessary to impose periodicity $\chi \sim \chi + \frac{\pi L}{r_0}$ in the compactified dimension. The resulting geometry has no horizon, it looks like a cigar with a tip at $r = r_0$. The most stable configuration which satisfies this constraint, the so called AdS Soliton [18], is fully specified by the metric,

$$ds^2 = L^2 \frac{dr^2}{f(r)} + r^2(-dt^2 + dx^2 + dy^2) + f(r)d\chi^2 \quad (1)$$

with $f(r) = r^2 - \frac{r_0^4}{r^2}$. In this background we introduce a charged scalar field Ψ minimally coupled to a Maxwell field, which leads to the following five-dimensional Einstein-Maxwell-scalar gravity theory,

$$S = \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |\nabla_\mu \Psi - iq A_\mu \Psi|^2 - m^2 |\Psi|^2 \right). \quad (2)$$

We refer to (2), together with the metric (1), as the AdS soliton holographic superconductor [11]. We note that this model, unlike the usual AdS-Schwarzschild superconductor, has a well defined zero temperature limit.

The properties of this model, investigated in [17] in the probe limit and in [21] including the backreaction of the scalar on the metric, depend on the value of the chemical potential μ , which corresponds to the time component of the gauge field at the boundary (see below). For $\mu < \mu_c$ the conductivity in the linear response regime vanishes and therefore the field theory resembles that of a Mott insulator [17]. However for $\mu > \mu_c$ the scalar condenses and the system becomes a superconductor though the background is still an AdS Soliton. It is therefore expected that some of the Mott insulator physics might still be at play in this regime. For sufficiently large $\mu \gg \mu_c$ a confinement-deconfinement transition occurs for any finite temperature but not at strictly zero temperature [21]. Superconductivity survives this transition but the gravity background becomes an AdS black hole [12].

We restrict ourselves to the probe limit in which the backreaction of the gauge field and scalar on the metric is negligible. Several reasons justify this choice. It was recently shown in [22] that the insulator-superconductor transition in the AdS Soliton does not proceed from a dynamical instability. Therefore it is not in principle possible, even in a calculation including backreaction, to induce the transition by a quench. Moreover the results of [21], including backreaction, at zero temperature are similar to those obtained in the probe limit [17].

The equations of motion (EOM).-

Since we are interested in time-evolution we seek solutions of the EOM that depend on both the temporal and radial coordinates: $A = (A_t, A_r, 0, 0, 0)$, $\Psi = |\Psi|e^{i\theta}$ where $A_t, A_r, |\Psi|$ and θ are functions of t and r [23]. In the following, we will work with the gauge-invariant quantities $M = A - d\theta$. Without losing generality we set $L = r_0 = q = 1$ and $m^2 = -15/4$ so that the time evolution depends only on the chemical potential. For numerical purposes it is more convenient to make the following change of variables, $z = 1/r$ and $|\psi| = |\Psi|/r^{3/2}$ so that the boundary is at $z = 0$. The resulting EOM are:

$$\begin{aligned} 0 &= |\psi| \left(-(z^4 - 1) M_z^2 - z^4 M_t^2 + 9z^6/4 \right) + 4z^7 \partial_z |\psi| \\ &\quad + (z^4 - 1) z^4 \partial_z^2 |\psi| + z^4 \partial_t^2 |\psi|, \quad (3) \\ 0 &= (z^4 - 1) z \partial_t \partial_z M_z + (z^4 + 3) \partial_t M_z + (3z^6 + z^2) \partial_z M_t \\ &\quad + (z^4 - 1) z^3 \partial_z^2 M_t + 2z^4 M_t |\psi|^2, \quad (4) \\ 0 &= \partial_t^2 M_z + 2z M_z |\psi|^2 + z^2 \partial_t \partial_z M_t, \quad (5) \\ 0 &= -(z^4 - 1) z |\psi| \partial_z M_z + z^3 |\psi| \partial_t M_t + 2z^3 M_t \partial_t |\psi| \\ &\quad - 2M_z \left(z(z^4 - 1) \partial_z |\psi| + (z^4 + 1) |\psi| \right). \quad (6) \end{aligned}$$

We note that Eqs. (4), (5) and (6) are not independent since Eq.(6) can be obtained from Eqs. (4) and (5).

Boundary conditions.-

In order to solve the EOM above it is necessary to specify initial conditions ($t = 0$) and boundary conditions at the tip $z = 1$ and at the boundary $z = 0$. Near the boundary and with the definitions above,

$$\begin{aligned} |\psi| &= |\psi_1(t)| + |\psi_2(t)|z + \mathcal{O}(z^2), \\ M_t &= \mu(t) - \rho(t)z^2 + \mathcal{O}(z^3), \\ M_z &= \mathcal{O}(z^3) \end{aligned} \quad (7)$$

where μ and ρ stand for the chemical potential and the charge density of the dual field theory respectively. We choose solutions for which $|\psi_1| \equiv 0, |\psi_2| \neq 0$. Following the standard AdS/CFT dictionary [12] the condensation of the operator \mathcal{O}_2 ,

$$\langle \mathcal{O}_2(t) \rangle = |\psi_2(t)| \quad (8)$$

stands for the order parameter of the dual field theory which describes the time evolution of the strongly-coupled superconductor. Both the gauge field and the scalar must not be singular at the tip. A simple choice is $|\psi| = a_1 + a_2(1-z) + \dots$, $M_t = a_3 + a_4(1-z) + \dots$ and $M_z = a_5 + a_6(1-z) + \dots$, where the a_i 's depend only on time. In order to obtain the initial conditions, $t = 0$, we solve the EOM in the static limit for $\mu(t=0) \equiv \mu_i$.

QNM of the scalar field.-

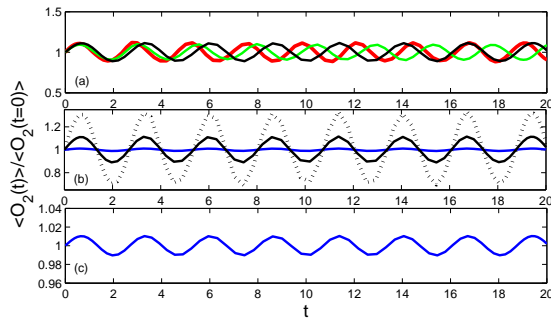


FIG. 2. (Color online) $\langle \mathcal{O}_2(t) \rangle$ (8) for different quenches, defined by $\mu_f, \mu_i > \mu_c$. In full agreement with the QNM results (see Fig. 1) only undamped oscillations are observed. (a) $\mu_i = 3, \mu_f = 3.2$ (red), $\mu_i = 2.6, \mu_f = 2.7$ (green), $\mu_i = 1.89, \mu_f = 2.0$ (black). The frequency of the oscillations is controlled by μ_i . (b) $\mu_i = 1.89$ and $\mu_f = 1.90$ (blue), 2.0 (black), and 2.3 (dotted). The amplitude of $\langle \mathcal{O}_2(t) \rangle$ increases with $|\mu_f - \mu_i|$. (c) Blown up of $\mu_i = 1.89, \mu_f = 1.90$ in (b). The oscillations are small but clearly visible. These results indicate that there exist strongly interacting field theories that after a quench do not thermalize.

In order to gain insight on the asymptotic dynamics of $\langle \mathcal{O}_2(t) \rangle$ we compute the QNM associated to the scalar $|\psi|$. The QNM's in the gravity theory correspond to the poles of the retarded Green's function of the dual field theory [19]. Therefore it is a powerful tool to investigate the asymptotic time evolution of the order parameter after a small perturbation. We define $|\psi(t, z)| =$

$|\psi_0(z) + e^{-i\omega t} \tilde{\psi}(z)|$ where $\tilde{\psi}$ is small with respect to the static solution $\psi_0(z)$. We carry out a similar expansion for the other fields. We keep only terms in the EOM that are leading in these small perturbations. The QNM are defined as the discrete set of complex,

$$\omega(\mu) = \omega_R - i\omega_I \quad (9)$$

frequencies that solve the EOM. A zero QNM is a signature of a phase transition [25]. As was expected, see Fig. 1, it occurs at $\mu = \mu_c$ where the insulator-superconductor transition occurs [17].

For $\mu < \mu_c$ (region I), the EOMs decouple and the relevant QNMs correspond to the pole of the retarded Green's function of the scalar. As shown in Fig. 1, the QNM's only contain a real part with $\omega_R \propto |\mu - \mu_c|$.

For $\mu > \mu_c$, in the superconducting phase, the calculation is more challenging since the scalar is coupled to a gauge field. In order to compute the QNM's we use Chebyshev spectral method where the perturbations above are explicitly gauge-invariant. This is important in order to avoid spurious QNM's. In Fig. 1 we plot the two lowest order QNM's ω as a function of μ . The black dot is the marginal stable mode which corresponds to $\omega = 0$, this is the critical point. For all $\mu > \mu_c$, $\omega_I = 0$

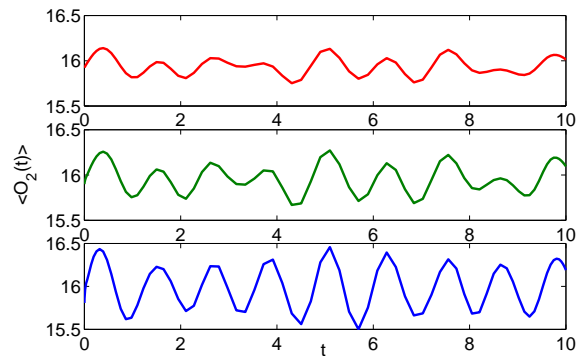


FIG. 3. (Color online) $\langle \mathcal{O}_2(t) \rangle$ (8) for $\mu_i = 5.0$ and, from top to bottom, $\mu_f = 5.04, 5.07, 5.14$. For any $\mu_i > 3.8$ and sufficiently small quenches oscillations of $\langle \mathcal{O}_2(t) \rangle$ become increasingly chaotic. For smaller μ 's the second order QNM cannot be excited and oscillations are not chaotic.

and $\omega_R \neq 0$ which indicates undamped oscillations of $\langle \mathcal{O}_2(t) \rangle$. This is a striking indication that a holographic superconductor in a AdS Soliton background does not thermalize. As can be observed in Fig. 1 there is a jump of the QNM at $\mu = \mu_c$. We believe that this reflects the effect in the superconducting phase of the mass gap, which according to [17], is related to Mott physics. Region II is also characterized by a growing (decreasing) lowest (second) QNM's as μ increases. This trend is reversed for $\mu \geq 3.8$ (region III). In the next section we shall see that precisely around these values there is a qualitative change in the oscillating pattern of the order parameter, from simple sinusoidal to more complex oscillations. In order to fully confirm this lack of thermalization we study $\langle \mathcal{O}_2(t) \rangle$ next.

Full time evolution of $\langle \mathcal{O}_2 \rangle$ after a quench.-

QNM's only provide information about the dynamics for asymptotically long times and for small perturbations. In order to study shorter times and stronger perturbations we compute explicitly the full time evolution of $\langle \mathcal{O}_2(t) \rangle$ (8) after a quench of the chemical potential $\mu(t)$ at $t \approx 0$. This type of quench has a clear physical interpretation: on the field theory side, a larger μ indicates a stronger order parameter [12]. A larger amplitude of $\langle \mathcal{O}_2(t) \rangle$ indicates a stronger interaction. Therefore by changing abruptly the chemical potential we are also modifying the effective interaction. At least qualitatively this is exactly the type of quench which is commonly used in condensed matter [16].

For numerical reasons the quench cannot be instantaneous. We assume that $\mu(t)$ goes from $\mu(0) = \mu_i$ to $\mu(\tau) \approx \mu_f$ in a time $\tau = 0.5/\mu_i$ (time is measured in units of μ_i) which is much smaller than the relaxation time of the system. The exact form of the time dependence for $0 < t < \tau$ is not important. For the sake of simplicity the quench is defined by imposing an additional boundary condition at $\mu(\tau) = \mu_f$. The latter is not strictly a quench as the effective value of the interaction strength changes smoothly on time in order to verify the boundary condition at $t = \tau$ however it is a good approximation since $\tau \sim 0.5 \ll t_f \sim 20$.

The EOM (4), (5) and (6) with boundary conditions (7) and the quench definition above were solved by using spectral methods. We discretize the EOM on a two dimensional Chebyshev grid with 100 points along the t direction and 20 points along z direction. We study times up to $t_f = 20$. Technically it is feasible to go up to $t_f = 40$ by adding more points along the t direction but the results are very much the same as those for $t_f = 20$. In Fig. 2 and Fig. 3 we plot $\langle \mathcal{O}_2(t) \rangle$ (8) for different quenches strength parametrized by $\mu(t=0) \equiv \mu_i$ and $\mu(t=\tau) \equiv \mu_f$. We explore a broad range of parameters with the only restriction that $\mu_i, \mu_f > \mu_c$. Several comments are in order: a) undamped simple sinusoidal oscillations are observed for $\mu_i < 3.8$, b) the frequencies of the oscillations increase with μ_i , c) the amplitude is an increasing function of $|\mu_i - \mu_f|$. This is expected as a stronger quench indicates a larger difference between the initial and the final coupling, d) similar results are obtained by different types of quenches and different choices of τ , e) similar results are also observed for weakly coupled superconductors [16] in a certain region of parameters, f) for $\mu_i > 3.8$ and small quenches we observe (see Fig. 3) a more complex oscillating pattern. As the quench strength increases the simple sinusoidal

pattern corresponding to $\mu_i < 3.8$ is gradually recovered. This minimum $\mu \sim 3.8$ is the one for which the lowest two order QNM's contribute simultaneously to the order parameter dynamics. In summary thermalization never occurs in our strongly coupled superconductor. This is the main result of the paper.

As was mentioned previously the probe limit is enough because the transition is not induced by a dynamical instability [22] and because previous results including backreaction [21] were qualitatively similar. As a further verification we have reproduced in the probe limit the main findings of [9] in the broken phase of the AdS black hole background which included the backreaction of the scalar on the metric. Finally we stress that our results are valid provided that quantum gravity corrections are neglected. For sufficiently long times it is expected that these corrections will induce some sort of thermalization in the system. However, even if thermalization finally occurs, its typical equilibration time is much larger than in other field theories with a gravity dual [4, 6].

In conclusion, we have investigated the time evolution of a holographic superconductor by computing the QNM's and the explicit time evolution of the order parameter after a quench. Our main result is that the order parameter does not thermalize. It performs undamped sinusoidal oscillations typical of insulators or integrable systems which do not reach thermal equilibrium after a quench. We have also identified a region, related to the existence of relatively close QNM's, for which the oscillations of the order parameter follow an intricate pattern. Since the dual field theory has a mass gap we speculate that the physical mechanism that prevents thermalization is related to localization caused by interactions.

Note: At the time we were finalizing this work a paper [26] was posted on arXiv which also studies a quench in an AdS Soliton background.

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